MATH 207(c)–Analysis IV

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1. For the following functions, determine the nature of the singularity at $z = z_0$ (i.e. regular point, pole or essential singularity), compute the residue, calculate the Laurent series and determine the radius of convergence.

(a) $f(z) = \frac{z^2+2z+1}{z^2-1}$, $z_0 = 1$.

(b) $f(z) = z^2 \sin(\frac{1}{z})$, $z_0 = 0$.

(c) $f(z) = e^{\frac{1}{z}} \sin(\frac{1}{z})$, $z_0 = 0$.

2. Consider the function

$$f(z) = \frac{\sin(z^2+1)}{(z^2+1)^2}.$$

(a) Find all the singularities of f and determine their nature.

(b) Compute the residue at each singularity.

(c) Determine the radius of convergence of the Laurent series around each singularity.

3. Consider the function

$$f(z) = \frac{\sin(z)}{(z+1)(z-2)(z^2+1)}.$$

(a) Find all the singularities of f and determine the order of the poles.

(b) Let $\gamma(t) = 10e^{it}$, $t \in [0, 2\pi]$. Compute the integral

$$\int_\gamma f(z)\, dz.$$

4. Consider the function

$$f(z) = \frac{1}{z^4-1}.$$

(a) Find all the singularities of f and determine their nature.

(b) Compute the integral

$$\int_{\gamma_r} f(z)\, dz$$

for any value of $r \neq 1$, where γ_r is the circle of radius r centered at the origin and oriented counter-clockwise.

5. For the following functions, compute the *singular* part of their Laurent series at $z = z_0$ and determine the radius of convergence of the (full) Laurent series.

(a) $f(z) = \dfrac{\sin(z)}{\sin(z^2)}$, $z_0 = 0$.

(b) $f(z) = \dfrac{1}{\cos(\frac{\pi}{2}z)}$, $z_0 = 1$.

(c) $f(z) = \dfrac{\log(1+z)}{\sin(z^2)}$, $z_0 = 0$.

(d) $f(z) = \dfrac{\sin(z)}{z \cdot (e^z - 1)}$, $z_0 = 0$.

6. *(The nature of essential singularities.)* In this exercise, we will consider the function $f(z) = e^{\frac{1}{z}}$.

(a) Show that $f(z)$ has an essential singularity at $z_0 = 0$.

(b) Show that in any neighborhood of $z_0 = 0$, f attains every value of $\mathbb{C}^*$ infinitely often. More formally, you have to show the following: For any $R > 0$ and any $y \in \mathbb{C}$, there exist infinitely many z's with $|z| < R$ such that $f(z) = y$. *(Hint: If w_0 satisfies $e^{w_0} = y$, what are the other w's satisfying $e^w = y$?)*

Remark. In general, if a function f has an essential singularity at $z = z_0$, then in any neighborhood of z_0 the function will attain all values in $\mathbb{C}$ with the exception of at most one, infinitely many times. This is known as Picard's great theorem.

(c) Show that the above cannot be true for a function $g(z)$ with a pole at $z = 0$: In this case, show that, for $R > 0$, the function g restricted on the disc $B_R(0)$ only takes values satisfying $|g(z)| \geqslant R_1(R)$ with $R_1(R) \to +\infty$ as $R \to 0$.